8-24 Midterm Monday Ch 13 (from this class)
Ch 16.1-16.3

Ex: (cont.)

Show $\vec{F} = (e^x\cos y + yz)\hat{i} + (xz - e^x\sin y)\hat{j} + (xy+z)\hat{k}$

is conservative over its natural domain and find a potential function.

↖ where should this function make sense (here, $D=\mathbb{R}^3$)

Ex: $\vec{F} = \sqrt{x}\,\hat{i} + y^2\hat{j} + \hat{k}$

(here, D = upper half space $x \geq 0$, $(y,z)\in\mathbb{R}^2$)

Soln: $\nabla \times \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \vec{F}$

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(xy+z) - \frac{\partial}{\partial z}(xz - e^x\sin y)\right)\hat{i}$$

$$- \left(\frac{\partial}{\partial x}(xy+z) - \frac{\partial}{\partial z}(e^x\cos y + yz)\right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x}(xz - e^x\sin y) - \frac{\partial}{\partial y}(e^x\cos y + yz)\right)\hat{k}$$

Saying $\operatorname{curl}(\vec{F}) = 0 \Rightarrow$ each component is 0, for example

$$\frac{\partial}{\partial y}(xy+z) - \frac{\partial}{\partial z}(xz - e^x\sin y) = 0$$

$$\frac{\partial}{\partial y}(xy+z) = \frac{\partial}{\partial z}(xz - e^x\sin y)$$

All partials continuous and $\operatorname{curl}\vec{F} = 0$ $\Rightarrow$ conservative

$$= (x-x)\hat{i} - (y-y)\hat{j} + (-e^x\sin y + z - (-e^x\sin y + z))\hat{k}$$
$$= \vec{0}$$

① $\dfrac{\partial f}{\partial x} = e^x \cos y + yz$

② $\dfrac{\partial f}{\partial y} = xz - e^x \sin y$

③ $\dfrac{\partial f}{\partial z} = xy + z$

not depend upon y, z

Integrate:

$$f(x,y,z) = \int \frac{\partial f}{\partial x}\,dx \overset{①}{=} e^x \cos y + xyz + g(y,z)$$

Now use equation 2:

$$xz - e^x \sin y \overset{②}{=} \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^x \cos y + xyz + g(y,z)\right)$$

$$= xz - e^x \sin y + \frac{\partial g}{\partial y}$$

$$\Rightarrow \quad \frac{\partial g}{\partial y} = 0$$

Integrate $\int \frac{\partial g}{\partial y}\,dy$

$$\Rightarrow \quad g(y,z) = h(z)$$

since constant with respect to y

$$\Rightarrow \quad f(x,y,z) = e^x \cos y + xyz + h(z)$$

$$xy + z \overset{③}{=} \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(f)$$

$$= xy + \frac{\partial h}{\partial z}$$

$$\Rightarrow \quad \frac{\partial h}{\partial z} = z$$

Integrate

$$\Rightarrow \quad h(z) = \frac{z^2}{2} + C$$

Finally a true constant

$$\Rightarrow \boxed{f(x,y,z) = e^x \cos y + xyz + \frac{z^2}{2} + C}$$

Easy to check: ∇f is easy to calculate.

Alternative: Ansatz "educated guess"

Guess $f(x,y,z)$ then compute all partials to show that it works.

Risky strategy: if you're wrong, could waste time (and I can't give that much partial credit for it)

... unless you're correct.

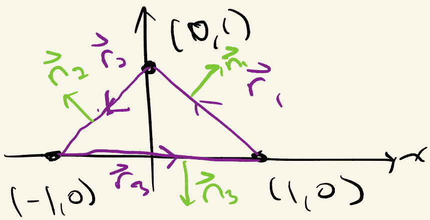
Ex:

$\vec{F}(x,y) = (x+y)\hat{i} - (x^2+y^2)\hat{j}$

Find the flux out of the triangle with vertices $(1,0)$, $(0,1)$, and $(-1,0)$.

Soln:

Sketch



$\vec{n} = \vec{T}\times\hat{k}$ want pointing out

parametrize CCW for outward flux

$$\vec{r}_1(t) = \underbrace{\hat{i}}_{\text{starting}} + t\underbrace{(\hat{j}-\hat{i})}_{\text{Ending}-\text{Starting}}$$

$$= (1-t)\hat{i} + t\hat{j} \qquad 0 \le t \le 1$$

$$\vec{r}_2(t) = \hat{j} + t(-\hat{i}-\hat{j})$$

$$= -t\hat{i} + (1-t)\hat{j} \qquad 0 \le t \le 1$$

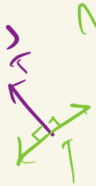
$$\vec{r}_3(t) = t\hat{i} \qquad -1 \le t \le 1$$

check parametrization:
plug in $t=a$, $t=b$

$\vec{r}_1(0) = (1-0)\hat{i} + 0\hat{j}$
$= \hat{i}$

$\vec{r}_1(1) = (1-1)\hat{i} + 1\hat{j}$
$= \hat{j}$

Now, for normal vectors. One definition of $\vec{n}$ is $\vec{n} = \vec{T}\times\hat{k}$. But it's usually easier to realize $\vec{n}$ is perpendicular to $\vec{T}$ (and so perpendicular to $\vec{r}\,'(t)$), then pick outward facing



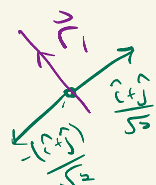
$$\vec{r}_1'(t) = -\hat{i}+\hat{j} \quad \Rightarrow \quad \vec{n}_1 \cdot \vec{r}_1'(t) \overset{\text{want}}{=} 0$$

$$(a_1\hat{i}+b_1\hat{j})\cdot(-\hat{i}+\hat{j}) = 0$$

$$-a_1 + b_1 = 0$$

$$\Rightarrow a_1 = b_1$$

unit vector $\rightarrow$ $\Rightarrow \vec{n}_1 = \pm\dfrac{(\hat{i}+\hat{j})}{\sqrt{2}}$



pick + because outward facing.

$\vec{r}_1(t) = (1-t)\hat{i} + t\hat{j}$ $\quad 0 \le t \le 1$ $\qquad \vec{r}_1{}'(t) = -\hat{i} + \hat{j}$ $\qquad \vec{n}_1 = \dfrac{\hat{i}+\hat{j}}{\sqrt{2}}$

$\vec{r}_2(t) = -t\hat{i} + (1-t)\hat{j}$ $\quad 0 \le t \le 1$ $\qquad \vec{r}_2{}'(t) = -\hat{i} - \hat{j}$ $\qquad \vec{n}_2 = \dfrac{-\hat{i}+\hat{j}}{\sqrt{2}}$

$\vec{r}_3(t) = t\hat{i}$ $\quad -1 \le t \le 1$ $\qquad \vec{r}\,'(t) = \hat{i}$ $\qquad \vec{n}_3 = -\hat{j}$

$\vec{F}(x,y) = (x+y)\hat{i} - (x^2+y^2)\hat{j}$

$$\int_{C_1} \vec{F}\cdot\vec{n}\,ds = \int_0^1 \vec{F}(\vec{r}_1(t))\cdot\vec{n}_1(t)\,|\vec{r}_1{}'(t)|\,dt$$

$$= \int_0^1 \left((1-t+t)\hat{i} - ((1-t)^2+t^2)\hat{j}\right)\cdot\frac{(\hat{i}+\hat{j})}{\cancel{\sqrt{2}}}\cdot\cancel{\sqrt{2}}\,dt$$

$$= \int_0^1 \left[1 - (1-2t+2t^2)\right]dt$$

$$= \frac{1}{3}$$

Alternative: Since C is CCW and outward normal,

$$\int_{C_1} \vec{F}\cdot\vec{n}\,ds = \int_{C_1} M\,dy - N\,dx$$

$$= \int_0^1 1\cdot dt - \left(-(1-t)^2-t^2\right)(-dt)$$

$$= \int_0^1 dt - \left(1-2t+2t^2\right)dt$$

$$= \frac{1}{3}$$

$M = (x+y)$
$= ((1-t)+t)$
$= 1$

$N = -(x^2+y^2)$
$= -((1-t)^2+t^2)$
$= -(1-t)^2 - t^2$

$dx = d(1-t)$
$= -dt$

$dy = d(t)$
$= dt$

Note - didn't need to calculate $\vec{n}$! (works ~~i~~ll in 3D)

$$\int_{C_2} \vec{F}\cdot\vec{n}_2\,ds = \int_0^1 \underbrace{\left((1-2t)\hat{\imath} - \left(t^2+(1-t)^2\right)\hat{\jmath}\right)}_{\vec{F}(\vec{r}_2(t))} \cdot \underbrace{\frac{(-\hat{\imath}+\hat{\jmath})}{\sqrt{2}}}_{\vec{n}_2} \underbrace{|\vec{r}_2\,'(t)|\,dt}_{ds}$$

$$= \int_0^1 2t-1-t^2-(1-t)^2\,dt$$

$$= -\frac{2}{3}$$

$$\int_{C_3} \vec{F}\cdot\vec{n}_3\,ds = \int_{-1}^1 (t\hat{\imath}-t^2\hat{\jmath})\cdot(-\hat{\jmath})\cdot|\vec{r}_3\,'(t)|\,dt$$

$$= \int_{-1}^1 t^2\,dt$$

$$= \frac{2}{3}$$

$$\Rightarrow \oint_C \vec{F}\cdot\vec{n}\,ds = \int_{C_1} \vec{F}\cdot\vec{n}_1\,ds + \int_{C_2} \vec{F}\cdot\vec{n}_2\,ds + \int_{C_3} \vec{F}\cdot\vec{n}_3\,ds$$

$$= \frac{1}{3} \quad -\frac{2}{3} \quad +\frac{2}{3}$$

$$= \frac{1}{3}$$

Ex: (related to ps7 #4)

Let C be unit circle, oriented CCW.

Let $f(x,y)=e^{-(x^2+y^2)}$

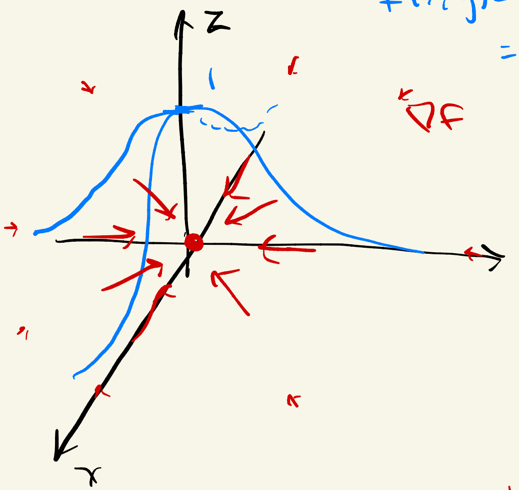
Compute and sketch ∇f, and calculate the flux of ∇f over C

Note: $\oint_C \nabla f \cdot d\vec{r} = 0$ since ∇f is conservative + loop property

(equiv: $\oint_C \nabla f \cdot d\vec{r} = f(a) - f(a)$
$= 0$
by fund thm. line int)

Soln:

$f(x,y) = e^{-(x^2+y^2)}$
$= e^{-r^2}$



Gradient points in direction of steepest ascent, and size of gradient reflects magnitude

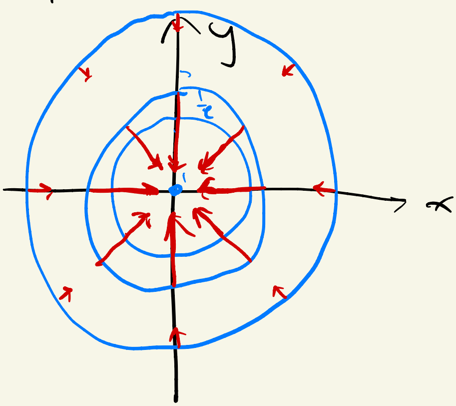
$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$= -2x e^{-(x^2+y^2)}\hat{i} - 2y e^{-(x^2+y^2)}\hat{j}$$

How to analyze:

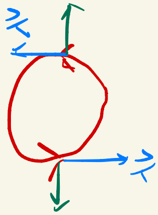
$$\nabla f = \underbrace{2e^{-(x^2+y^2)}}_{\text{radially dependent scalar } > 0}\left(-x\hat{i} - y\hat{j}\right)$$

points towards origin



Flux $= \oint_C \vec{F}\cdot\vec{n}\,ds$

unit normal for circles are easy:



CCW parametrization

$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$

$\vec{r}\,'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$

Need: $\vec{n} \perp \vec{r}\,'(t)$ and outward unit vector

unit

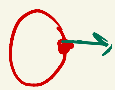
$$(a\hat{i} + b\hat{j})\cdot(-\sin t\,\hat{i} + \cos t\,\hat{j}) = 0$$

$$\vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

$$\Rightarrow -a\sin t + b\cos t = 0$$

$$\Rightarrow a = \pm\cos t$$
$$b = \pm\sin t$$

will work. and luckily, $\vec{n}$ is already a unit vector.
To pick outward normal (pick + or −), plug in a point and check:

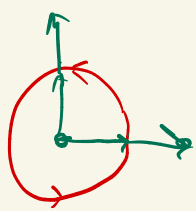


t=0

if $t=0$

$$\vec{n} = \pm 1\,\hat{i} \pm 0\,\hat{j}$$

want: $+\hat{i}$ direction $= \hat{i}$



$$\Rightarrow \vec{n} = \cos t\,\hat{i} + \sin t\,\hat{j}$$

notice: $\vec{n} = \vec{r}$

this always works for CCW circles (as long as $\vec{n}$ is normalized to be a unit vector.)

$$\text{Flux} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{n}(t) \cdot |\vec{r}\,'(t)|\, dt$$

$$= \int_0^{2\pi} e^{-((\cos t)^2 + \sin^2 t)} \left(-2\cos t\,\hat{i} - 2\sin t\,\hat{j}\right) \cdot \left(\cos t\,\hat{i} + \sin t\,\hat{j}\right) dt$$

$$= \int_0^{2\pi} e^{-1} (-2) \left(\cos^2 t + \sin^2 t\right) dt$$

$$= \int_0^{2\pi} \frac{-2}{e} \cdot 1 \, dt$$

$$= -\frac{4\pi}{e}$$

Ex:

Work integral of $\vec{F} = -y\hat{i} + z\hat{j} + 2x\hat{k}$

for C: $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$

$0 \le t \le 2\pi$

(helix)

Soln:

Could try to find potential ... but notice this $-y$ and z business, which reminds us of whirlpool ($\vec{F} = -y\hat{i} + x\hat{j}$). Check $\hat{k}$ component of curl:

$$\frac{\partial}{\partial y}(-y) \stackrel{?}{=} \frac{\partial}{\partial x}(z)$$

$$-1 \neq 0$$

Nope! No fund. thm, this isn't conservative

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt$$

$$= \int_0^{2\pi} \left(-\sin t\,\hat{i} + t\hat{j} + 2\cos t\,\hat{k}\right)\cdot\left(-\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}\right)dt$$

Alt: $\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy + P\,dz$

$$= \int_0^{2\pi} \sin^2 t + t\cos t + 2\cos t\,dt$$

full period!

$$= \left[\frac{t}{2} - \cancelto{0}{\frac{\sin t\cos t}{2}} + t\cancelto{0}{\sin t} + \cancelto{0}{\cos t} - \cancelto{0}{2\sin t}\right]_0^{2\pi}$$

$$= \frac{2\pi}{2} = \pi$$